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LINES OF CURVATURE ON MINIMUM DEVELOPABLES

BY FREDERICK S. WOODS

IN the treatises on Differential Geometry are usually found one or both of the two following theorems, different in form but the same in content:

1. If every line upon a surface is a line of curvature, the surface is a plane or a sphere; and conversely.

2. If for any surface the fundamental quantities of the first order are in a constant ratio to those of the second order, the surface is a plane or a sphere; and conversely.

These theorems are true for real surfaces; but in general no restriction to real surfaces is made either in stating or proving the theorems, although the proofs are generally given with the use of explicit or suppressed hypotheses which rule out of consideration one class of imaginary surfaces, namely the developable surfaces which contain the imaginary circle at infinity. Accordingly it seems to have escaped notice that these surfaces, which we shall call minimum developables, should be named with the plane and the sphere in the enunciation of the above theorems.

We define a line of curvature on a surface as any curve such that the normals to the surface taken along the curve form a developable surface. Since we do not restrict ourselves to real quantities, the geometric terms employed have analytic definitions. A surface is defined by the equation

$$z=f(x, y),$$

where $f(x, y)$, together with its derivatives of at least three orders, is finite and continuous within the (x, y)-region considered. We write, as usual,

$$p=\frac{\partial z}{\partial x}, \quad q=\frac{\partial z}{\partial y}, \quad r=\frac{\partial^2 z}{\partial x^2}, \quad s=\frac{\partial^2 z}{\partial x \partial y}, \quad t=\frac{\partial^2 z}{\partial y^2}.$$

The tangent plane at any point is defined by the equation

$$p(X-x)+q(Y-y)-(Z-z)=0,$$

and the normal line by the equations

$$\frac{X-x}{p}=\frac{Y-y}{q}=\frac{Z-z}{-1}.$$

The necessary and sufficient condition that the surface should be a minimum developable is

$$p^2 + q^2 + 1 = 0,$$

and the above equations show that the normal line then lies in the tangent plane and is in fact a generator of the minimum developable. It is clear then that any line on the minimum developable satisfies the definition of a line of curvature, for the normals to the surface form a developable, namely the minimum developable itself.

The differential equation of the lines of curvature is

$$[pqr - s(1+p^2)]dx^2 + [r(1+q^2) - t(1+p^2)]dxdy \\ + [pqt - s(1+q^2)]dy^2 = 0,$$

and consequently all lines upon a surface are lines of curvature when and only when

$$\begin{aligned} pqr - s(1+p^2) &= 0, \\ r(1+q^2) - t(1+p^2) &= 0, \\ pqt - s(1+q^2) &= 0. \end{aligned} \qquad (1)$$

An easy calculation shows that these conditions are satisfied if the surface is a plane, a sphere, or a minimum developable. Conversely we shall show that the above relations lead necessarily to one of these three surfaces.

(1) If $r = s = t = 0$, equations (1) are satisfied, and integration gives a plane

$$z = ax + by + c.$$

Further, if any one of the three quantities, r, s, t, is zero, the others are also, as may readily be seen.

(2) Assuming $r \pm 0$, $s \pm 0$, $t \pm 0$, we may write the equations (1) in the form

$$\frac{pq}{s} = \frac{1+p^2}{r} = \frac{1+q^2}{t} = \phi,$$

or

$$pq = s\phi, \qquad (2)$$

$$1 + p^2 = r\phi, \qquad (3)$$

$$1 + q^2 = t\phi, \qquad (4)$$

where ϕ is an unknown function of x and y.

These equations being satisfied by all points on the surface, we have necessarily

$$rq + ps = \phi \frac{\partial s}{\partial x} + s\frac{\partial \phi}{\partial x}, \tag{5}$$

$$2ps = \phi\frac{\partial r}{\partial y} + r\frac{\partial \phi}{\partial y}, \tag{6}$$

which are obtained by differentiating (2) with respect to x and (3) with respect to y. Remembering that $\frac{\partial s}{\partial x} = \frac{\partial r}{\partial y}$, we obtain from (5) and (6),

$$rq - ps = s\frac{\partial \phi}{\partial x} - r\frac{\partial \phi}{\partial y}. \tag{7}$$

Similarly, by differentiating (2) with respect to y, (4) with respect to x, and subtracting one result from the other, we find

$$-sq + pt = -t\frac{\partial \phi}{\partial x} + s\frac{\partial \phi}{\partial y}. \tag{8}$$

Hence, from (7) and (8),

$$p(rt - s^2) = -(rt - s^2)\frac{\partial \phi}{\partial x},$$

$$q(rt - s^2) = -(rt - s^2)\frac{\partial \phi}{\partial y},$$

and therefore either

$$(2a) \qquad rt - s^2 = 0,$$

or

$$(2b) \qquad \frac{\partial \phi}{\partial x} = -p, \quad \frac{\partial \phi}{\partial y} = -q.$$

(2*a*) The equation $rt - s^2 = 0$, is the condition that a surface should be a developable surface. From (1) we have also

$$1 + p^2 + q^2 = 0.$$

The developable is therefore a minimum developable.

(2*b*) The equations

$$\frac{\partial \phi}{\partial x} = -p, \quad \frac{\partial \phi}{\partial y} = -q$$

give at once

$$\phi = c - z.$$

This value substituted in (2), (3), and (4) gives relations which are readily transformed into

$$\frac{\partial}{\partial x}(pz - cp) = \frac{\partial}{\partial y}(qz - cq) = -1,$$

$$\frac{\partial}{\partial y}(pz - cp) = \frac{\partial}{\partial x}(qz - cq) = 0;$$

whence

$$pz - cp = a - x,$$

$$qz - cq = b - y;$$

and therefore

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = d^2,$$

the equation of a sphere.

We see then that equations (1) lead necessarily to a plane, a sphere, or a minimum developable, and we have already seen that each of these surfaces furnishes a true solution of the equations.

In considering the second of the theorems stated at the beginning of this article, we need to notice a difference of usage in regard to the fundamental quantities of a surface. If the equations of the surface are

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

there is essential agreement in using as the fundamental quantities of the first order

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

Gauss introduced as fundamental quantities of the second order

$$D = A\frac{\partial^2 x}{\partial u^2} + B\frac{\partial^2 y}{\partial u^2} + C\frac{\partial^2 z}{\partial u^2},$$

$$D' = A\frac{\partial^2 x}{\partial u \partial v} + B\frac{\partial^2 y}{\partial u \partial v} + C\frac{\partial^2 z}{\partial u \partial v},$$

$$D'' = A\frac{\partial^2 x}{\partial v^2} + B\frac{\partial^2 y}{\partial v^2} + C\frac{\partial^2 z}{\partial v^2},$$

where

$$A = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}, \quad B = \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}, \quad C = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial v}.$$

Subsequent writers have used these quantities, divided by $\sqrt{EG - F^2}$.

Now the necessary and sufficient condition that the surface should be a minimum developable is that $EG - F^2 = 0$, and hence for such a surface the fundamental quantities of the second order as last defined are meaningless. We shall accordingly use Gauss's definitions. The problem is to find surfaces for which

$$\frac{E}{D} = \frac{F}{D'} = \frac{G}{D''}.$$

If we place $u = x$, $v = y$, as we may without loss of generality, these equations become equations (1) already discussed.

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